The scalar self-force for generic extreme-mass-ratio orbits in a Kerr spacetime

Zachary Nasipak

13 April 2017

The University of North Carolina at Chapel Hill

Outline

Motivation and Background

Black Hole Perturbation Theory

Solving for the Scalar Self-Force

Results

Conclusions

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Compact binaries and gravitational wave astronomy

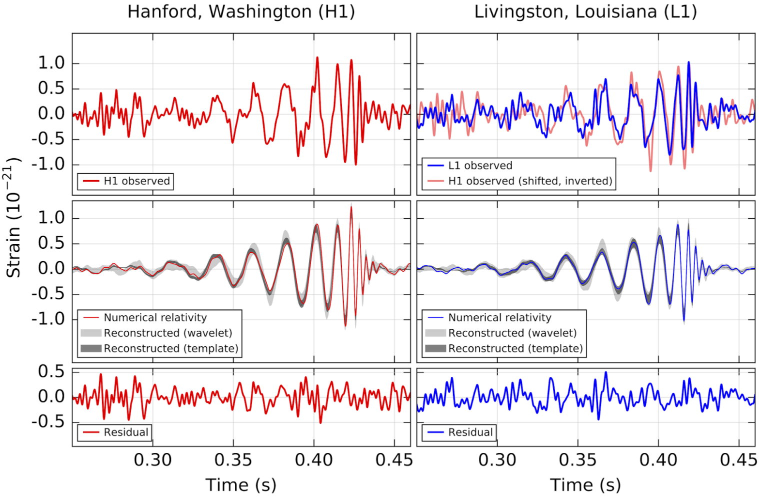

Image Credit: LIGO Collaboration

Future gravitational wave detectors:

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2024:

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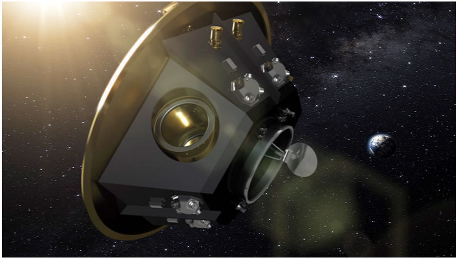

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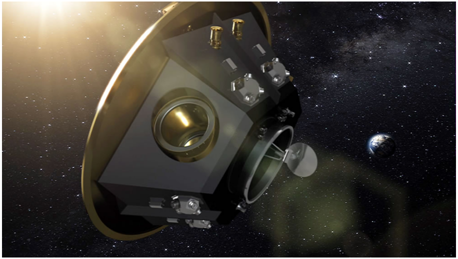

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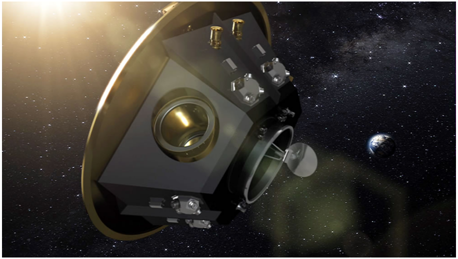

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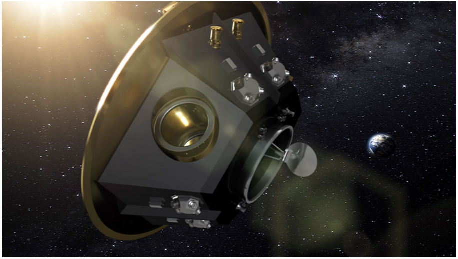

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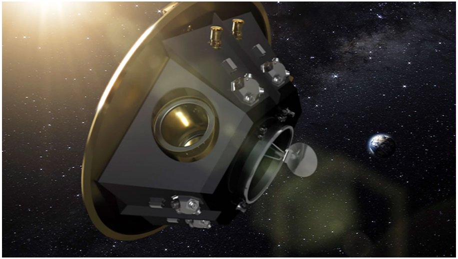

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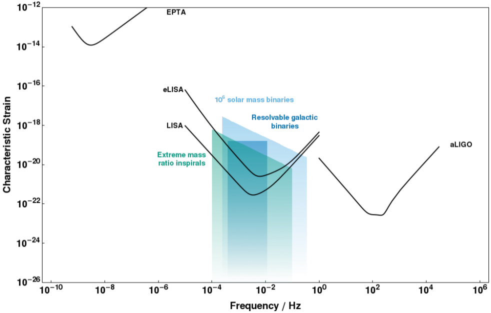
- Observe mergers out to $z \sim 20$

What do we expect to observe?



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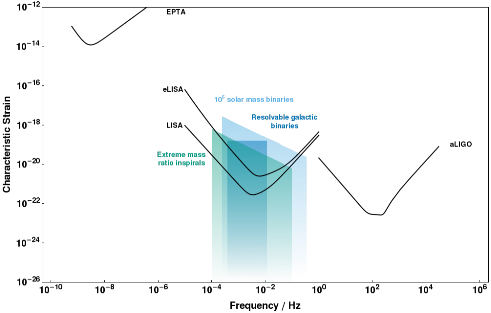

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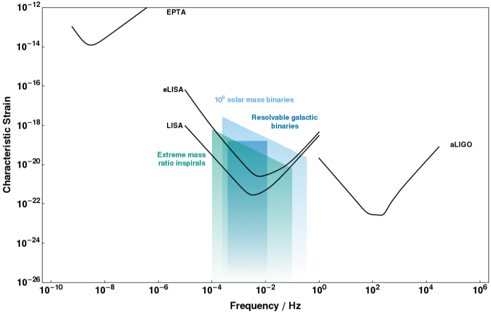

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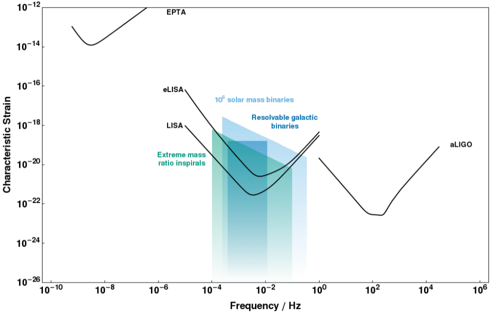

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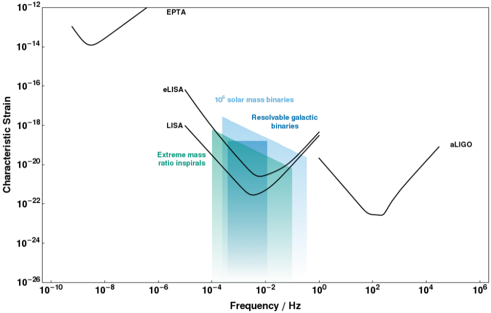

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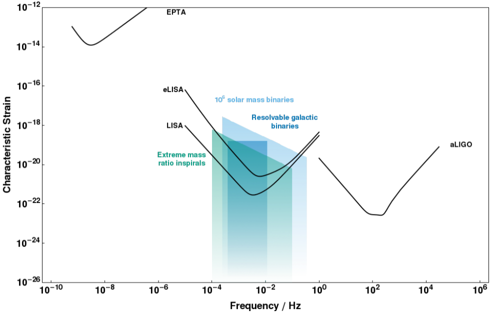

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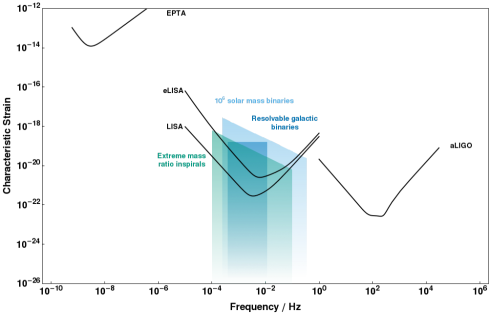

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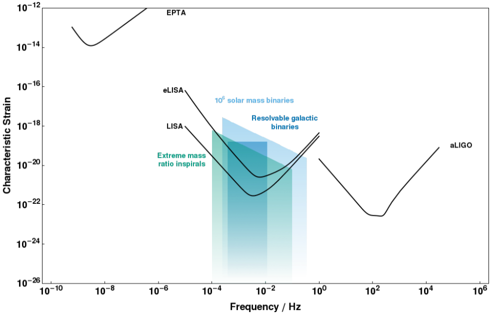

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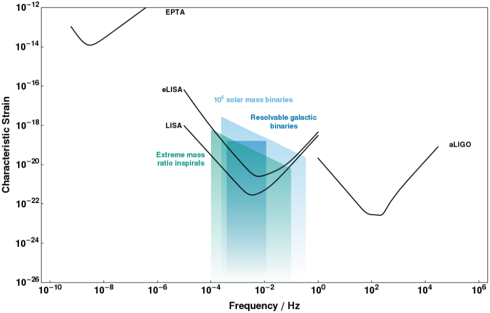

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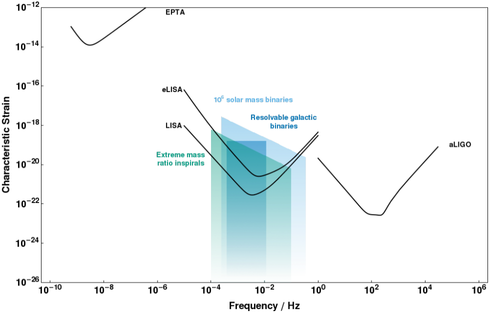

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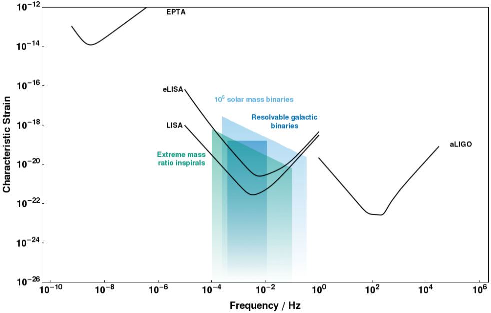

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See LISA White Papers!

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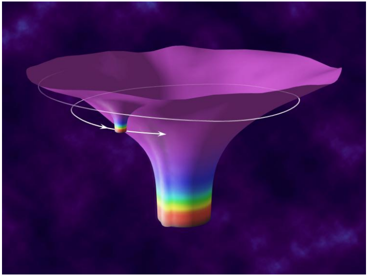

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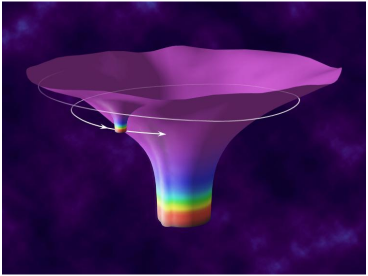

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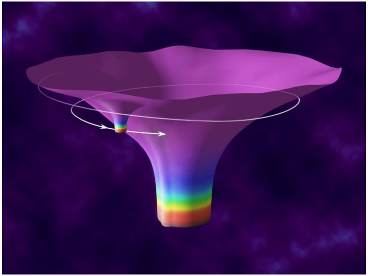

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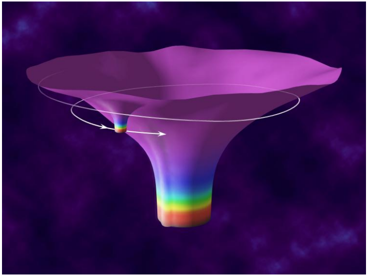

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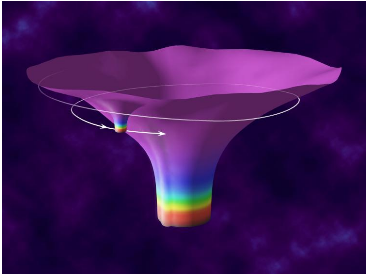

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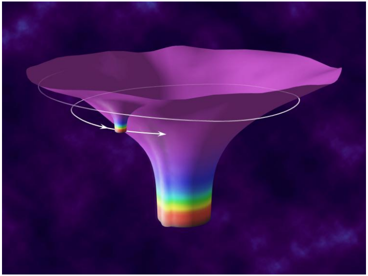

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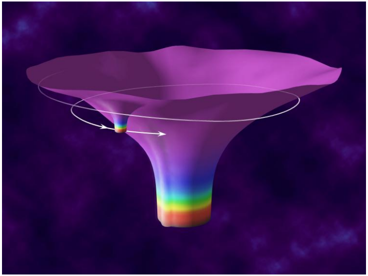

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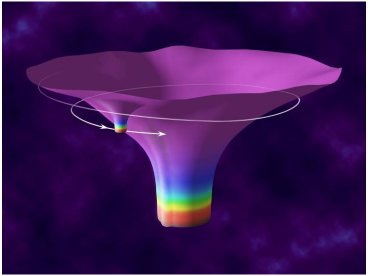

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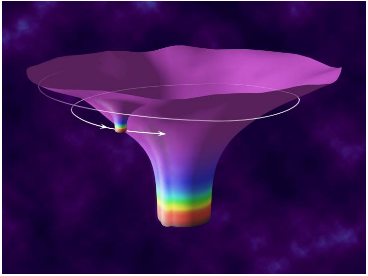

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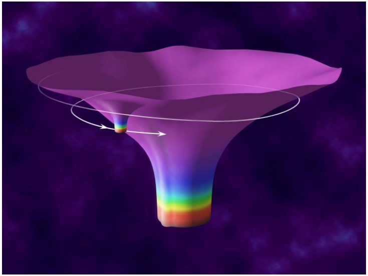

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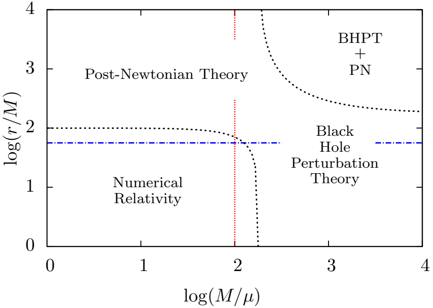

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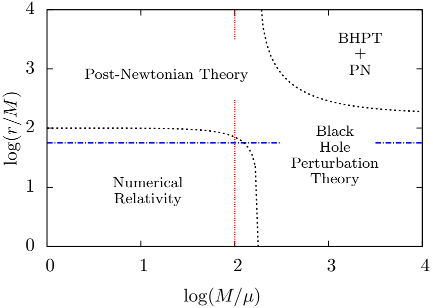

Computational Regimes of General Relativity

Leading approaches:

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  - widely separated binaries
- **Numerical Relativity**
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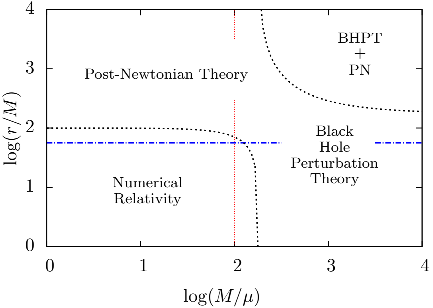

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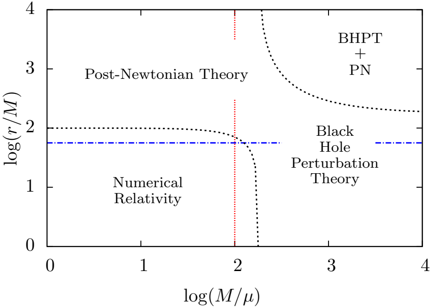

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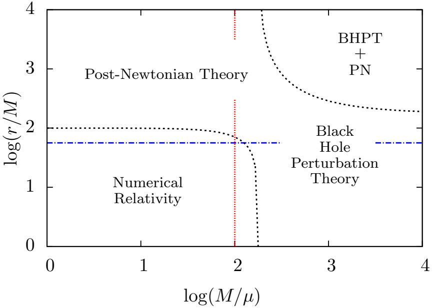

Outline

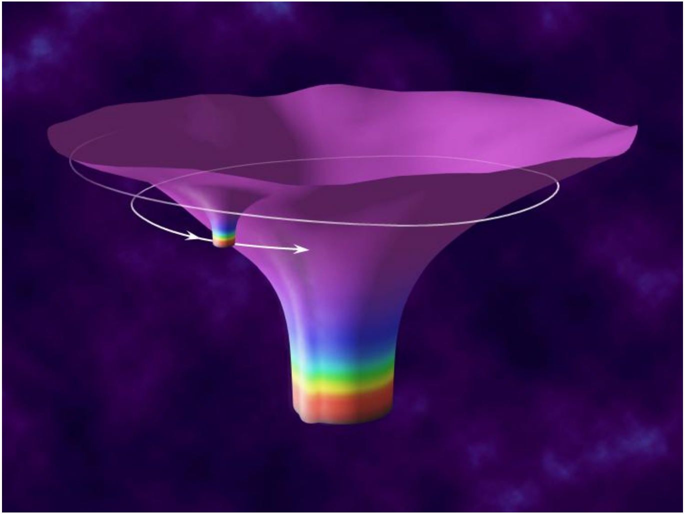

Image Credit: NASA

Black Hole Perturbation Theory (BHPT)

	Field Equations	Equations of Motion
0	$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$	$\frac{Du^\alpha}{d\tau} \equiv u^\beta \nabla_\beta u^\alpha = 0$
1	$\nabla_\alpha \nabla^\alpha \bar{h}_{\mu\nu} + 2R^\alpha{}_\mu{}^\beta{}_\nu \bar{h}_{\alpha\beta} = -16\pi T_{\mu\nu}$ $\nabla^\nu \bar{h}_{\mu\nu} = 0$	$\mu \frac{Du^\alpha}{d\tau} = F^\alpha[h_{\mu\nu}]$
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An Electron in Curved Space

DeWitt & Brehme (1960): e^- in orbit

- At first on a geodesic trajectory
- Over time, e^- orbit decays
- Bremsstrahlung-like radiation damping

Does this violate the Equivalence Principle?

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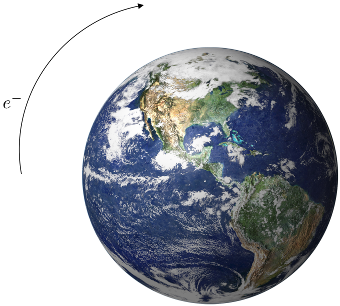
Object interacting with its own field known as *self-force* or *radiation-reaction*

Radiation-reaction due to body's own gravitational field $\Rightarrow$ *gravitational self-force (GSF)*



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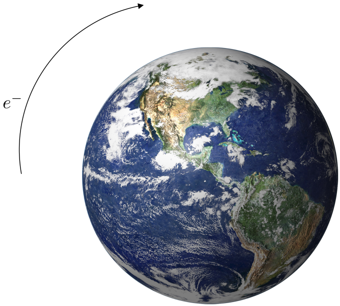

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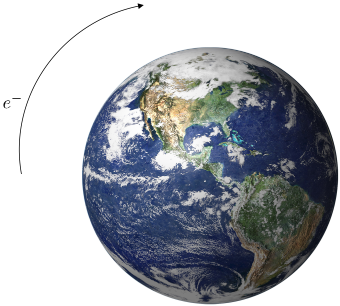

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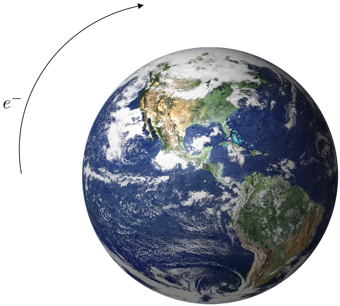

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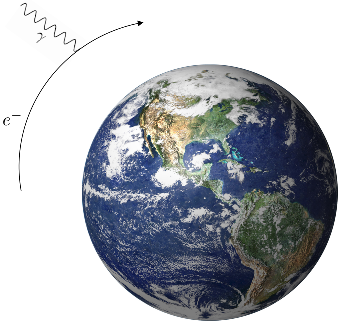

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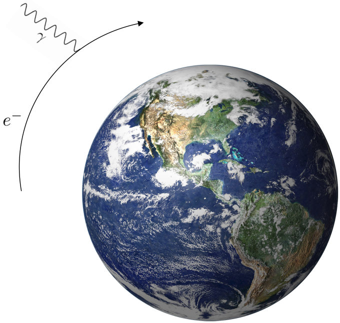

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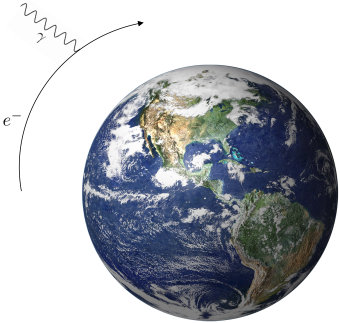

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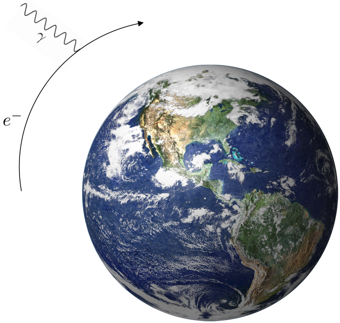

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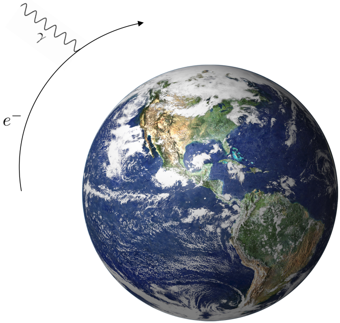

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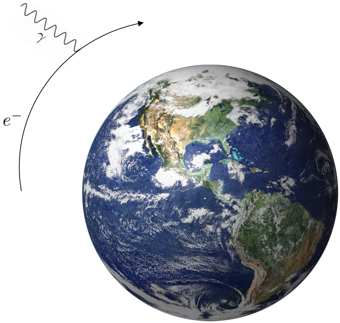

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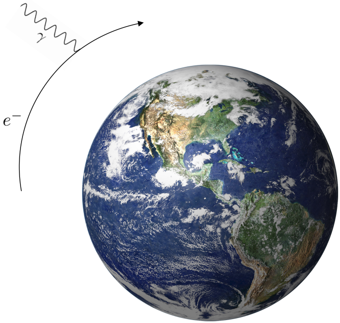

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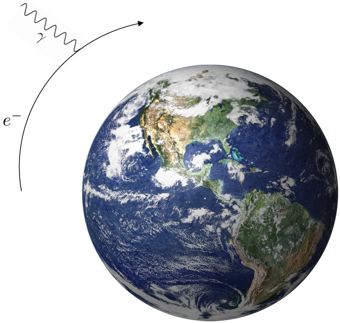

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Naive approach: particle still on geodesic

- Write out geodesic equations for full metric $g_{\mu\nu} + h_{\mu\nu}$

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- Force diverges along particle's trajectory
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A cautionary tale – BHPT and the point-particle limit

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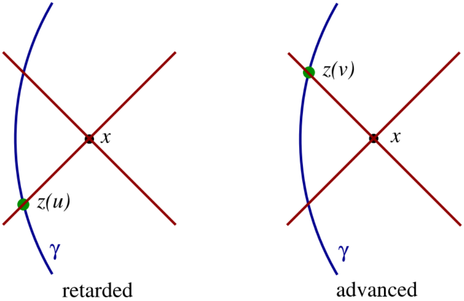

Image Credit: Poisson et. al LRR 14 (2011)

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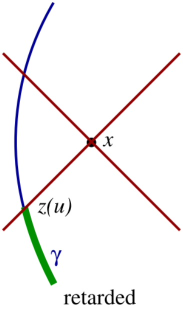

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For rigorous treatment of the GSF problem, see van de Meent **PRD 94** (2016)

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Motivation and Background

Black Hole Perturbation Theory

Solving for the Scalar Self-Force

Results

Conclusions

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Definition for SSF: $F^{\alpha}_{\text{SSF}} \equiv q\nabla^{\alpha}\Phi^{\text{R}}$

Solving for the F^{α}_{SSF}

1. Find the background geodesic motion on Kerr of scalar-charged particle
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2. Solve for the field using Klein-Gordon scalar wave-equation
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Use Boyer-Lindquist coordinates (t, r, θ, φ)

Solving the geodesic equations

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- Specify $\{p, e, \iota\} \to \{E, L_z, Q\}$ for bound motion
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  - Developed by Hopper et al (2015) for 200 digit calculations on Schwarzschild
  - Motion is periodic, has fundamental frequencies
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- With spectral integration, 100 digits of precision w/ ~ 500 samples

Background Geodesics on Kerr

Use Boyer-Lindquist coordinates (t, r, θ, φ)

Solving the geodesic equations

- 4 1st-order coupled ODEs
- Depend on Kerr parameters $\{a, M\}$ & orbital constants $\{E, L_z, Q\}$
- Specify $\{p, e, \iota\} \to \{E, L_z, Q\}$ for bound motion
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Constructing the Scalar Field

Field equation due to scalar charge source

$$\nabla_a \nabla^\alpha \Phi^{\text{ret}} = -4\pi\sigma \qquad \sigma = \sigma[z(\tau)]$$

Use mode decomposition and separation of variables

- Decompose field and source in frequency domain, then on the basis of *spheroidal harmonics* $S_{\hat{l}m\omega}(\theta, -\gamma^2)e^{im\varphi}$

$$\implies \qquad \left[\frac{d^2}{dr_*^2} + U_{\hat{l}mkn}(r)\right] X_{\hat{l}mkn}(r) = \tilde{\sigma}_{\hat{l}mkn}(r)$$

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Computational efficiency of SSI on Kerr

- $a/M = 0.5$, $e = 0.5$, $p = 15$, $\iota = \pi/3$
- Calculate $C^{\pm}_{2222}$ mode

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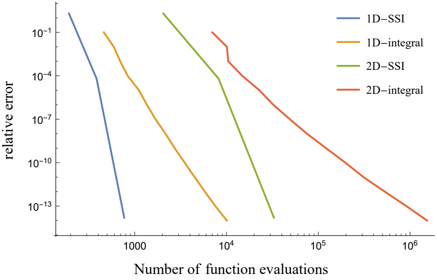

Image Credit: Thomas Osburn

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Decompose the regular field on basis of spherical harmonics

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$$F^{\mathrm{SSF}}_{\alpha} = \sum_{l=0}^{\infty} \left[\left(F^{l(\mathrm{ret})}_{\alpha}\right)^{\pm} - \left(F^{l(\mathrm{S})}_{\alpha}\right)^{\pm}\right]$$
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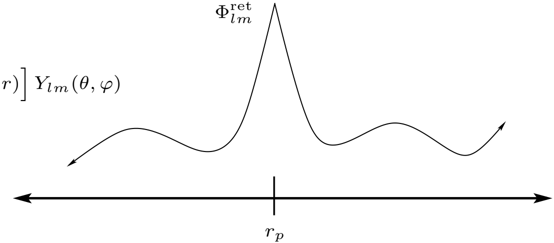

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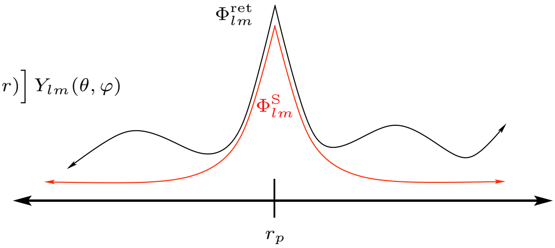

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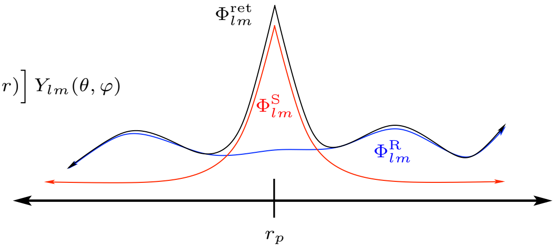

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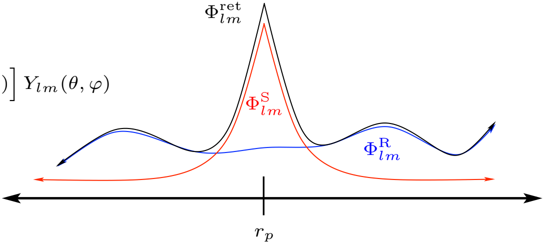

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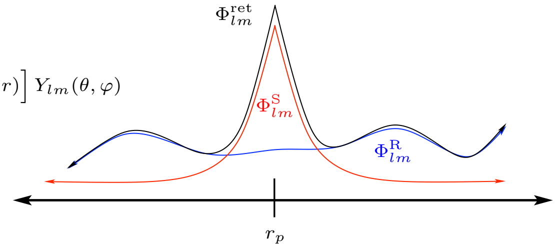

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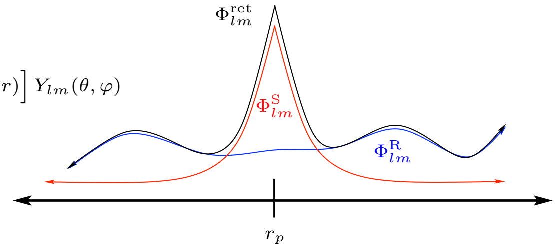

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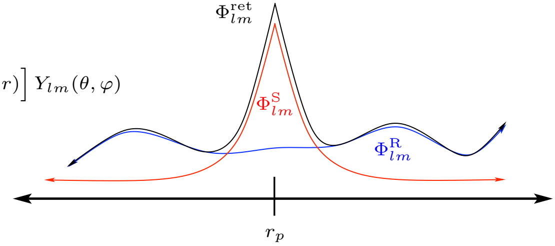

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$$= \sum_{l=0}^{\infty} \left[\left(F^{l(\mathrm{ret})}_{\alpha}\right)^{\pm} - A^{\pm}_{\alpha} L - B_{\alpha} - C_{\alpha}/L\right] - D_{\alpha} \qquad \text{w/ } L \equiv l + \frac{1}{2}$$

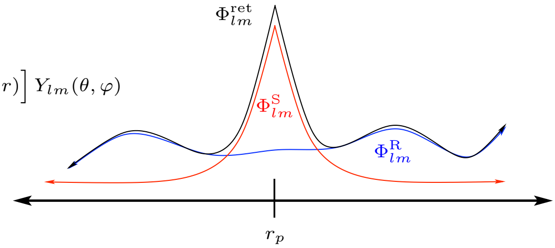
Mode-sum regularization valid for both GSF and SSF

Parameters analytically known for gravitational and scalar cases on Kerr

Mode-Sum Regularization

Decompose the regular field on basis of spherical harmonics

$$\Phi^{\mathrm{R}} = \sum_{lm} \Phi^{\mathrm{R}}_{lm}(t,r) Y_{lm}(\theta,\varphi)$$
$$= \sum_{lm} \left[\Phi^{\mathrm{ret}}_{lm}(t,r) - \Phi^{\mathrm{S}}_{lm}(t,r)\right] Y_{lm}(\theta,\varphi)$$



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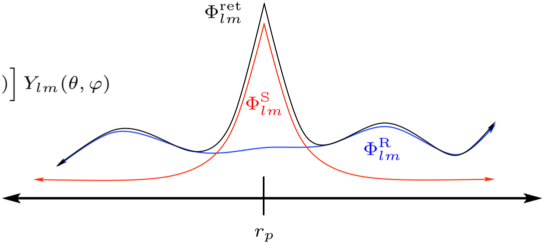
$$S_{\hat{l}m} e^{-im\varphi} = \sum_{l} b^{l}_{\hat{l}m} Y_{lm}(\theta,\varphi)$$



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Outline

Flux Code Validation in the Equatorial Plane

Fluxes on Kerr

- Gauge invariant quantities calculated by integrating components of the stress-energy at spatial infinity and the black hole horizon
- Conservation of energy ⇒ radiated energy = work done by SSF

Code and SSI Validation

$$\langle \dot{\mathcal{E}}^{\pm} \rangle = \sum_{lmkn} f_m(\omega_{mkn}) |C^{\pm}_{lmkn}|^2 \qquad \mathcal{E} \rightarrow E \text{ or } L_z$$

- Reference values in Warburton & Barack (2011):
  $p = 10$, $e = 0.2$, $a/M = -0.5$, $\iota = 0$

$$\langle \dot{E} \rangle^{\text{tot}} = 3.6565609775 \times 10^{-5}$$

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$$|1 - \langle \dot{E} \rangle^{\text{tot}} / \langle \dot{E} \rangle^{\text{ref}}| = 6.83703 \times 10^{-10}$$

$$|1 - \langle \dot{L}_z \rangle^{\text{tot}} / \langle \dot{L}_z \rangle^{\text{ref}}| = 3.13246 \times 10^{-10}$$

Flux Code Self-Consistency

Inclined orbit on Schwarzschild

- Spherical symmetry ⇒ physics should be unaffected by rotations

Equatorial case:

$p = 10$, $e = 0.2$, $a/M = 0$, $\iota = 0$

Inclined case:

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$\langle \dot{E} \rangle$ should have same value for both cases

$$\langle \dot{E} \rangle^{\text{inc}} = 3.21331398 \times 10^{-5}$$

$$|1 - \langle \dot{E} \rangle^{\text{inc}} / \langle \dot{E} \rangle^{\text{eq}}| = 2 \times 10^{-15}$$

SSF Code Validation in the Equatorial Plane

SSF of eccentric, equatorial orbit on Kerr

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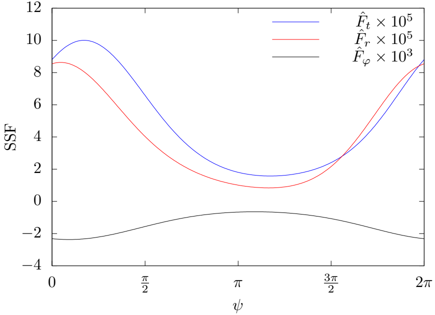

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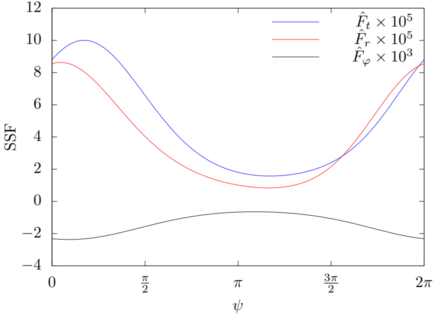

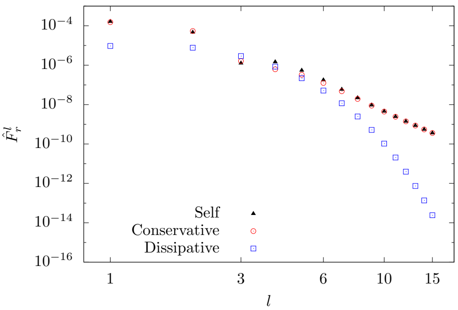

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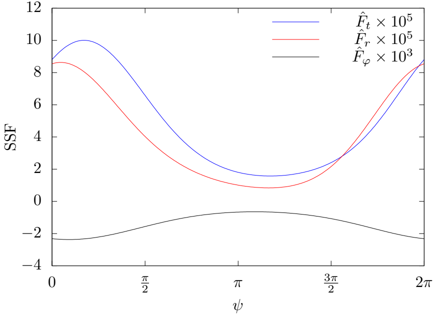
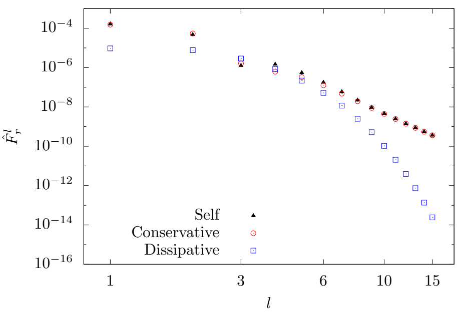
	$F_t \times M^2/q^2$	$F_r \times M^2/q^2$	$F_\varphi \times M/q^2$
disp	4.5×10^{-5}	9.3×10^{-6}	-2.4×10^{-4}
cons	2.1×10^{-5}	3.1×10^{-5}	-1.3×10^{-3}





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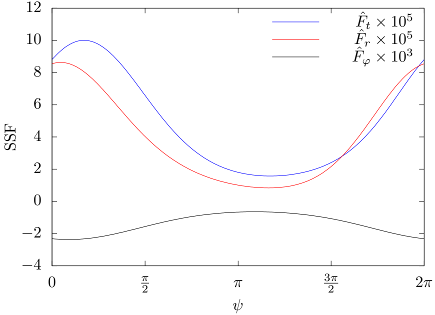
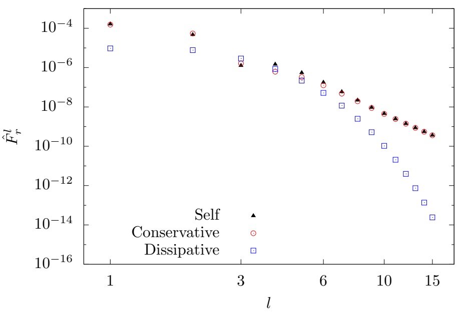
	$F_t \times M^2/q^2$	$F_r \times M^2/q^2$	$F_\varphi \times M/q^2$
disp	3.3×10^{-9}	7.3×10^{-9}	4.0×10^{-9}
cons	2.4×10^{-5}	2.0×10^{-4}	5.2×10^{-5}

Relative Error with Warburton & Barack (2011)





SSF Code Self-Consistency

SSF over an eccentric, inclined orbit in Schwarzschild limit

Rotate coordinate system from equatorial case

$$F_t(\iota) = F_t^{\text{eq}} \qquad F_\varphi(\iota) = F_\varphi^{\text{eq}} \cos\iota$$

$$F_r(\iota) = F_r^{\text{eq}} \qquad F_\theta(\iota) = \pm F_\varphi(\iota)\sqrt{\sec^2\iota - \csc^2\theta_p}$$

Specify orbital parameters $p = 10$, $e = 0.2$, $a/M = 0$, $\iota = \pi/3$

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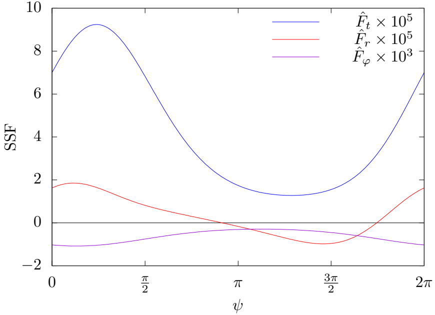

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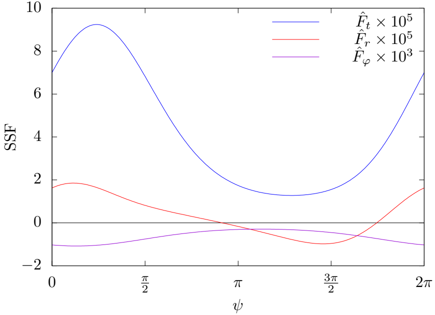

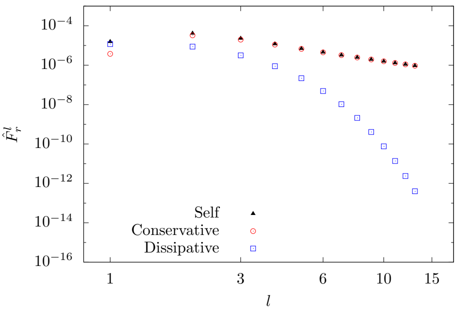

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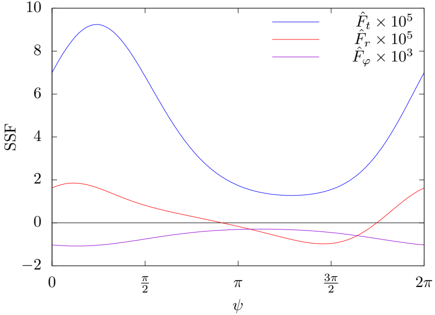
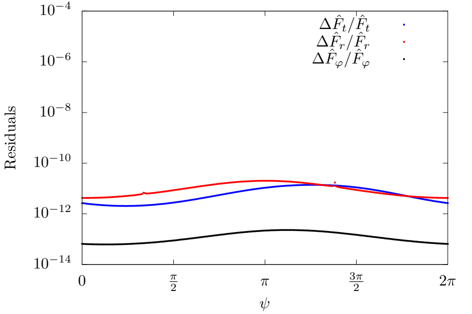
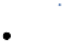




Fluxes for Inclined, Eccentric Orbit on Kerr

Inclined, eccentric orbit on Kerr: $p = 50$, $a/M = 0.5$, $e = 0.2$, $\iota = \pi/8$



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Total Energy and Angular Momentum Fluxes

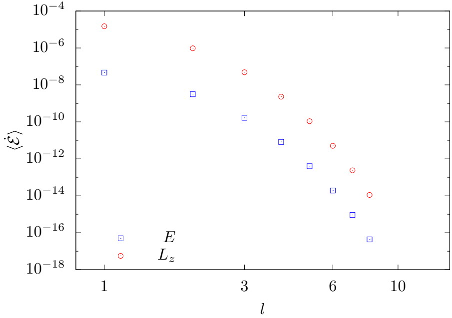
$\langle \dot{E} \rangle^{\text{tot}} = 5.01080253 \times 10^{-8}$ $\quad \langle \dot{L}_z \rangle^{\text{tot}} = 1.59571235 \times 10^{-5}$

$\langle \dot{E}^- \rangle = 1.19847920 \times 10^{-8}$ $\quad \langle \dot{L}_z^- \rangle = -2.3897897920 \times 10^{-8}$

$\langle \dot{E}^+ \rangle = 4.99881774 \times 10^{-8}$ $\quad \langle \dot{L}_z^+ \rangle = 1.598102137 \times 10^{-5}$

Cutoff of $l_{\text{max}} = 8$

$\Rightarrow$ relative accuracy $\sim$ 8 digits



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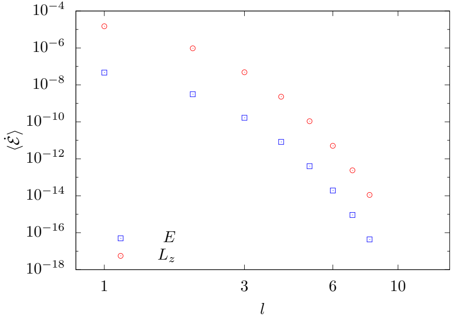

Outline

Motivation and Background

Black Hole Perturbation Theory

Solving for the Scalar Self-Force

Results

Conclusions

Conclusions

LISA to launch in 2034 - promising science

Success of LISA depends on success of theory

Black hole perturbation theory & the gravitational self-force (GSF) are promising mathematical formalisms for modeling EMRIs

Scalar self-force (SSF) is a powerful toy model

My SSF code

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- passes validation tests
- Provides first results of flux calculations on a generic orbit
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Thanks!

Dr. Charles Evans

(Advisor)

Dr. Thomas Osburn

(Collaborator)

NSF

(Funding)

Black Hole Perturbation Theory (BHPT)

	Field Equations	Equations of Motion
0	$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$	$\frac{Du^\alpha}{d\tau} \equiv u^\beta \nabla_\beta u^\alpha = 0$
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Gravitational Self-Force (GSF)

Naive approach: fictitious force

- Full spacetime metric given by $\mathrm{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$
- Self-force just fictitious "gravitational force" from describing motion with respect to background metric $\boldsymbol{g}$ instead of full metric $\mathbf{g}$

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Tail/Regular Field

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$$\nabla_\alpha \nabla^\alpha \Phi^{\text{ret}} = -4\pi\sigma$$

$$\nabla_\alpha \nabla^\alpha G(x, x') = -4\pi\delta^{(4)}(x, x')$$

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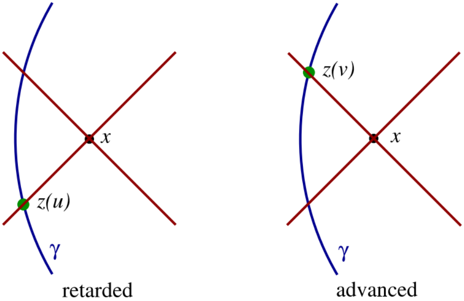

Image Credit: Poisson et. al LRR 14 (2011)

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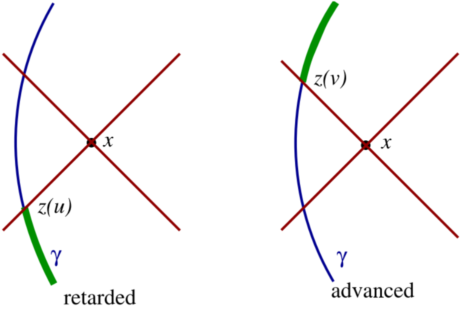

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Detweiler & Whiting (2003) $\implies \Phi = \Phi^{\rm S} + \Phi^{\rm R}$

Singular field: $\nabla_\alpha \nabla^\alpha \Phi^{\rm S} = -4\pi\sigma$

Regular field: $\nabla_\alpha \nabla^\alpha \Phi^{\rm R} = 0$

- Note that $\Phi^{\rm S}$ is time-symmetric

Naturally extends to gravitational case: $\Phi \to h_{\mu\nu}$

$\Rightarrow$ geodesic motion on $g_{\mu\nu} + h^{\rm (R)}_{\mu\nu}$

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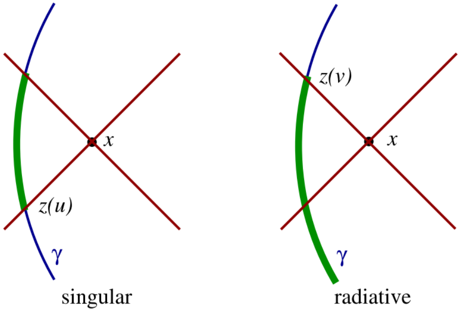

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Scalar Self-Force (SSF)

Recall definition of GSF: $F^{\alpha}_{\rm GSF} \equiv q\nabla^{\alpha\beta\gamma}h^{\rm R}_{\beta\gamma}$

Definition for SSF: $F^{\alpha}_{\rm SSF} \equiv q\nabla^{\alpha}\Phi^{\rm R}$

Equations of motion: $\mu u^{\beta}\nabla_{\beta}u^{\alpha} = F^{\alpha}_{\rm GSF} \quad \to \quad u^{\beta}\nabla_{\beta}(\mu u^{\alpha}) = F^{\alpha}_{\rm SSF}$

$$u^{\alpha}F^{\rm GSF}_{\alpha} = 0 \quad \to \quad u^{\alpha}F^{\rm SSF}_{\alpha} = -\frac{d\mu}{d\tau}$$

In practice, we calculate the physical, *retarded* field $\Phi^{\rm ret}$

$\implies$ regularization scheme

$$\begin{aligned} F^{\rm self}_{\alpha} &= q\nabla_{\alpha}\Phi^{\rm R} \\ &= q\nabla_{\alpha}(\Phi^{\rm ret} - \Phi^{\rm S}) \\ &= F^{\rm ret}_{\alpha} - F^{\rm S}_{\alpha} \end{aligned}$$

How to subtract two “infinities”?

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Computational problem at the turning points $r_{\min}$ & $r_{\max}$

$\Rightarrow$ parameterize r_p and θ_p

$$r_p(\psi) = \frac{pM}{1 + e\cos\psi} \qquad \cos\theta_p(\chi) = \sqrt{z_-}\cos\chi$$

$$z_- \equiv f(a, E, L_z, Q) = \cos^2\theta_{\min}$$

$$\frac{d\lambda}{d\psi} \equiv P^{(r)}(\psi) \qquad \frac{d\lambda}{d\chi} \equiv P^{(\theta)}(\chi)$$

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Computational problem at the turning points r_{min} & r_{max}

$\Rightarrow$ parameterize r_p and θ_p

$$r_p(\psi) = \frac{pM}{1 + e\cos\psi} \qquad \cos\theta_p(\chi) = \sqrt{z_-}\cos\chi$$

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Background Geodesics on Kerr

Line element for Kerr metric in Boyer-Lindquist coordinates (t, r, θ, φ)

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar\sin^2\theta}{\Sigma} dt d\varphi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\Sigma}\right)\sin^2\theta d\varphi^2$$

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Spectral orbit integration (SOI) theory

- Originally developed by Hopper et al (2015) for Schwarzschild
- Consider integrand $P^{(r)}(\psi)$: periodic, smooth, & C^∞
  - Represent as Fourier series
  - Coefficients fall off exponentially $\Rightarrow$ truncate after N coefficients
  - Truncated Fourier series is a bandlimited function
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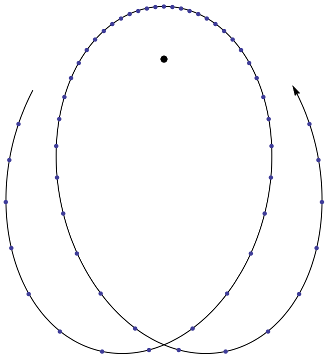
- Number of samples scales with accuracy goal



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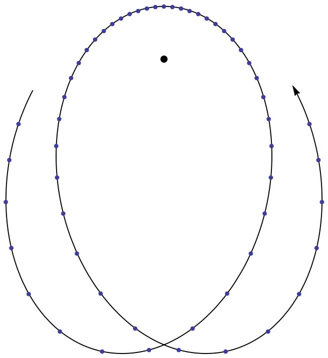

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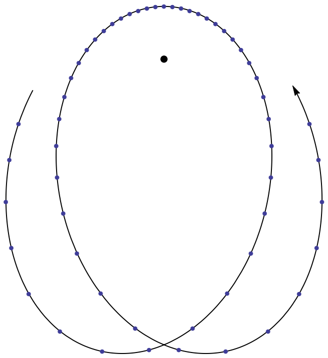

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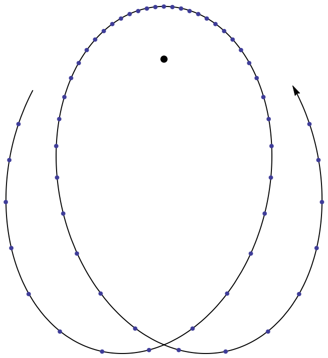

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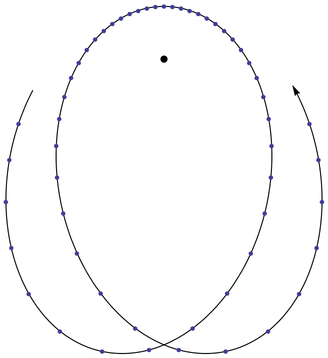

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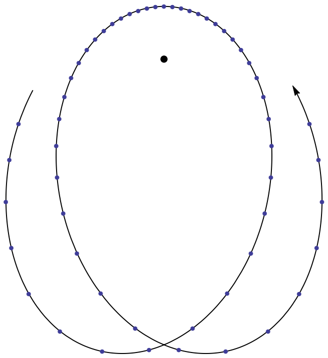

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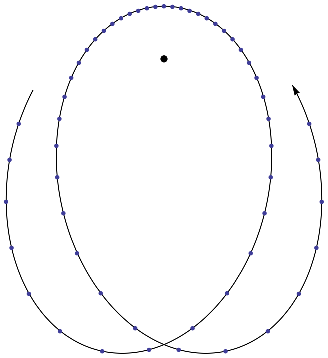

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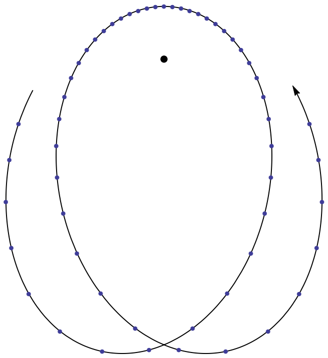

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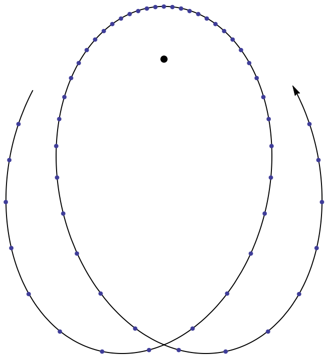

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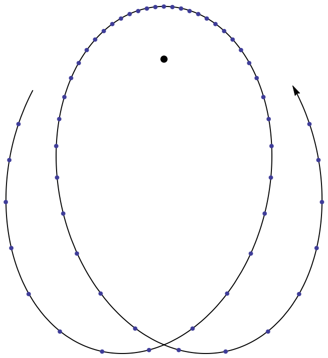

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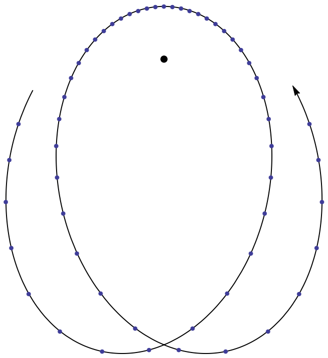

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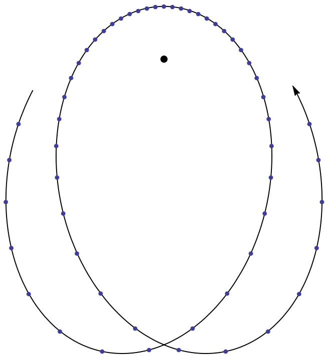

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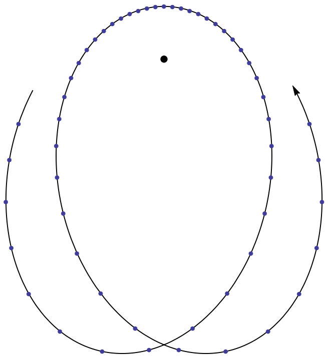

Sample SOI Calculation

$$\frac{d\lambda^{(r)}}{d\psi} = P^{(r)}(\psi)$$

$$P^{(r)}(\psi) = \sum_{j=0}^{\infty} \mathcal{P}_j^{(r)} \cos(j\,\psi)$$

$$\psi_i \equiv \frac{i\pi}{N-1} \qquad i \in 0, 1, \ldots, N-1$$

$$\mathcal{P}_j^{(r)} \simeq \frac{2}{N-1}\left[\frac{1}{2}P^{(r)}(0) + \frac{1}{2}(-1)^j P^{(r)}(\pi) + \sum_{i=1}^{N-2} P^{(r)}(\psi_i)\cos(j\psi_i)\right]$$

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Constructing the Scalar Field

Field equation due to scalar charge source

$$\nabla_a \nabla^\alpha \Phi^{\text{ret}} = -4\pi\sigma \qquad \sigma(t,r,\theta,\varphi) = q\int \delta(x - z(\tau))(-g)^{-\frac{1}{2}}\, d\tau$$

Use mode decomposition and separation of variables

- Decompose field and source in frequency domain, then on the basis of spheroidal harmonics $S_{\bar{l}m\omega}(\theta, -\gamma^2)e^{im\varphi}$

$$\Phi(t,r,\theta,\varphi) = \frac{1}{\sqrt{r^2+a^2}} \sum_{\bar{l}mkn} X_{\bar{l}mkn}(r) S_{\bar{l}mkn}(\theta) e^{im\varphi} e^{-i\omega_{mkn}t}$$

$$\sigma(t,r,\theta,\varphi) = -\frac{(a^2+r^2)^{3/2}}{4\pi\Sigma\Delta} \sum_{\bar{l}mkn} \tilde{\sigma}_{\bar{l}mkn}(r) S_{\bar{l}mkn}(\theta) e^{im\varphi} e^{-i\omega_{mkn}t}$$

$$\implies \qquad \left[\frac{d^2}{dr_*^2} + U_{\bar{l}mkn}(r)\right] X_{\bar{l}mkn}(r) = \tilde{\sigma}_{\bar{l}mkn}(r)$$

Solve the homogeneous radial equation w/ solutions $X^{\pm}_{\bar{l}mkn}$

MST Formalism

Compute $X^{\pm}_{\hat{l}mkn}$ using Mano-Suzuki-Takasugi (MST) function expansion formalism

- Limit $\omega \to 0$, horizon solutions $\to$ hypergeometric functions
  outer solutions $\to$ Coulomb wave functions
- Expand radial solutions into series of these functions

$$X^{\pm}(r) \sim f(r) \sum_n a_n^{\nu} p_{n+\nu}(r)$$

- Solving ODE for $X_{\hat{l}mkn}$ $\to$ solving algebraic equation for ν

Advantages

- Not as limited by machine precision as typical ODE solvers (e.g. Runge-Kutta)
- Nearly-analytic method

Disadvantages

- At high frequencies ν becomes imaginary
- At high frequencies, large cancellations in function summation $\Rightarrow$ large precision loss